Linear Systems of Quadrics and Regular Graded Clifford Algebras

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1 Introduction

This report will describe the results of a forthcoming paper. The proofs will be mostly omitted. We study regular graded Clifford algebras which are a particular class of Artin-Schelter regular algebras. These algebras have been classified in dimensions $n \leq 3$ by Artin, Tate, Schelter and Van den Bergh in [1], [2], [3], and there has been much work on dimension $n = 4$ but a complete classification is not known yet. The schemes of point modules of Artin-Schelter regular algebras are of vital importance in the classification in dimension 3 and it is expected that the schemes of point and line modules will play an equally important role in the classification in dimension four.

Regular graded Clifford algebras are a class of Artin-Schelter Regular algebras that can be described as Koszul duals of complete intersections of n quadrics in an n dimensional vector space V. They are given by $n(n-1)/2$ relations in $\mathrm{Sym}^2 V$ and any generic choice will produce an Artin-Schelter regular algebra. Lastly, a regular graded Clifford algebra A has a centre $k[W^*]$ with W^* a vector space of dimension n in degree two and A is a finitely generated module over its centre. The algebra A can be described as a graded Clifford algebra using the linear system of quadrics which give its Koszul dual $A^!$.

We study the space of linear modules and linear subalgebras of a regular graded Clifford algebra A. A linear module of dimension d is a cyclic A module with Hilbert Series $(1-t)^{-d}$. A linear subalgebra of dimension $n-d$ is a subalgebra which is a regular graded Clifford algebra of dimension $n-d$. We show that there is a natural bijective correspondence between these objects and the set of linear subspaces in V that are contained in a $n-d$ dimensional subspace of quadrics of the n dimensional system of quadrics that describes $A^!$.

The variety of codimension two linear modules of a generic regular graded Clifford algebra has a étale double cover which is a Calabi-Yau variety of dimension $n-2$. When $n = 4$ this variety is a Enriques surface known classically as a Reye Congruence.

Lastly we show that the space of regular graded Clifford algebras form a component of the moduli space of Artin-Schelter regular algebras of dimension $n \geq 4$.

2 Linear Systems of Quadrics

In this report we will be concerned with objects determined by a linear system of quadrics in a projective space of the same dimension. Let k be an algebraically closed field of characteristic zero. We will let $\mathbb{P}(V)$ be the projective space of *lines* in a vector space V of dimension n over the field k. Quadratic functions are given by elements of the vector space $\operatorname{Sym}^2 V^*$ and we will chose a linear subspace W of this vector space of dimension $n = \dim V$.

This subspace can be interpreted as a vector in $f \in \operatorname{Hom}(W, \operatorname{Sym}^2 V^*) \simeq W^* \otimes \operatorname{Sym}^2 V^*$. The variety $B = V(f = 0) \subset \mathbb{P}(W) \times \mathbb{P}(V)$ is a divisor of degree $(1, 2)$. Given w in W and v in V, the associated quadratic function is $w(v, v)$ and we will write S_w for the symmetric matrix associated to w, so $w(v, v) = v^T S_w v$. We will write Q_w for the quadric defined by $V(w(v, v) = 0) \subset \mathbb{P}(V)$, the fibre of the projection from B to $\mathbb{P}(W)$

We will write the linear system $|Q_w| = \mathbb{P}(W)$, and we will denote the base locus of a linear system $|D|$ by $\operatorname{Bs}|D|$. Since $W \subseteq \operatorname{Sym}^2 V^*$ we can let $A^! = k[V^*]/(W)$ be the homogeneous coordinate ring of the base locus of the linear system $\mathbb{P}(W)$. We will always assume that

$$\operatorname{Bs}|Q_w| = \bigcap_{w \in W} Q_w = \emptyset$$

$$\Leftrightarrow A^! = k[V^*]/(W) \text{ is a complete intersection.}$$

So $A!$ is a finite dimensional Koszul algebra ([6] Section 3.1) with Hilbert series

$$H_{A^!}(t) = \frac{(1-t^2)^n}{(1-t)^n} = (1+t)^n.$$

Our main object of study is the Koszul dual of $A^!$. Note that we can also consider $A^!$ to be a noncommutative ring with presentation

$$A^! = k\langle V^* \rangle / (W \oplus \wedge^2 V^*).$$

So its Koszul dual is given by

$$A = k\langle V \rangle / (W \oplus \wedge^2 V^*)^\perp.$$

Where $\perp$ indicates the orthogonal subspace with respect to the natural pairing between $V \otimes V$ and $V^* \otimes V^*$. Let $R = (W \oplus \wedge^2 V^*)^\perp$ be the relations of A.

Since $W \oplus \wedge^2 V^*$ certainly contains $\wedge^2 V^*$ we can conclude that $R \subseteq (\wedge^2 V^*)^\perp = \operatorname{Sym}^2 V$. The nondegenerate pairing between $V \otimes V$ and $V^* \otimes V^*$ induces a nondegenerate pairing between $A_2 = (V \otimes V)/R$ and $W \oplus \wedge^2 V^*$, giving a natural isomorphism $A_2 \simeq W^* \oplus \wedge^2 V$. We have exact sequences,

$$0 \to W \to \operatorname{Sym}^2 V^* \to A^!_2 \to 0$$

$$0 \to R \to \operatorname{Sym}^2 V \to W^* \to 0.$$

The Hilbert series of A is

$$H_A(t) = H_{A^!}(-t)^{-1} = (1-t)^{-n}$$

and $\dim R = n(n-1)/2$.

If we choose a basis w_i of W and let w^i be the corresponding dual basis, then we may also write the relations of A as

$$ab + ba = \sum w^i w_i(a, b)$$

for a, b in V and the w^i are central. So we can consider A to be the graded algebra generated by V, W^* where the elements of V have degree one and those of W^* have degree two. We can then eliminate the generators w^i in terms of elements of V. This gives the following Theorem, due to Kapranov [7].

Theorem 2.1 *Let A be a regular graded Clifford algebra. Then A is isomorphic to the algebra generated by $V \oplus W^*$ where V has degree one and W^* has degree two and is central, with the relations*

$$ab + ba = \sum w^i w_i(a, b)$$

where a, b are in V and w^i, w_i are dual bases of W^* and W.

We record some well known properties of the algebra A. Let $\nu_2(A) = \bigoplus A_{2n}$ be the second Veronese subalgebra of A. We may invert a set of homogeneous elements T of $k[W^*] \subset \nu_2 A$ and take degree zero fractions of $T^{-1}\nu_2(A)$. We get a sheaf of algebras $\mathcal{A}$ which is the sheaf of even Clifford algebras of the associated quadric bundle. Let $\Delta_i \subseteq \mathbb{P}(W)$ be the locus of quadrics with rank $\leq i$. So Δ_{n-1} is the locus of singular quadrics and is hypersurface of degree n in $|Q_w|$.

Proposition 2.2 *Suppose that A is a regular graded Clifford algebra. Then*

- *The centre of A is $k[W^*]$.*
- *A is a finitely generated modules over $k[W^*]$.*
- *The algebra $\nu_2(A)$ forms a sheaf of algebras $\mathcal{A}$ over $\mathbb{P}(W)$.*
- *$\mathcal{A}$ has rank 2^{n-1}*
- *As a vector bundle over $\mathbb{P}(W)$*

$$\mathcal{A} \simeq \bigoplus_{i=0}^{\lfloor n/2 \rfloor} \mathcal{O}(-i)^{\binom{n}{2i}}$$

- *If n is odd then*
  - *the centre of $\mathcal{A}$ is $\mathcal{O}_{\mathbb{P}(W)}$*
  - *$\mathcal{A}$ is Azumaya outside Δ_{n-1}.*
- *If n is even then*
  - *the centre of $\mathcal{A}$ is $\mathcal{O}_Z$ where $Z \to \mathbb{P}(W)$ is a double cover ramified on Δ_{n-1}.*
  - *A is Azumaya outside the preimage of Δ_{n-2} in Z.*

A graded algebra $A = \bigoplus n \geq 0$ with $A_0 = k$ and generated in degree one is *Artin-Schelter* regular of dimension d if

- finite global dimension d.
- $\mathrm{Ext}^i(k,k) = 0$ for $i \neq d$ and $Ext^d(k,k) = k$.

- it has polynomial growth $\dim A_n \leq cn^e$ for some e.

An algebra is Frobenius if A is isomorphic to its k linear dual as a left module ${}_AA \simeq_A A^*$.

The next Theorems are in Smith [12] and [9].

Theorem 2.3 *The algebra $A^!$ is Frobenius and A is Artin-Schelter regular if and only if $A^!$ is a complete intersection.*

3 Linear Modules and Subalgebras

Schemes of point modules are used to study Artin-Schelter regular algebras of dimension 3 in [2], [3]. Line schemes are studied in [10],[11]. We will look at linear modules of various dimensions.

A linear A-module N of dimension d is a cyclic A-module with Hilbert series

$$H_N(t) = 1/(1-t)^d.$$

So in particular we have

$$0 \to L \to A_1 = V \to N_1 \to 0$$

$$A_2 = W^* \oplus \wedge^2 V \twoheadrightarrow N_2 = M^*$$

So a linear module of dimension d gives a subspace $L \subseteq V$ such that $\dim L = n - d$ and a subspace $M \subseteq W^* \oplus \wedge^2 V$ of dimension $\dim M = (d+1)d/2$.

We define a linear projection of A of dimension $n-d$ to be a subalgebra B with Hilbert series

$$H_B(t) = 1/(1-t)^{n-d}$$

which is an Artin-Schelter regular algebra. In particular B will be generated in degree one by a subspace $B_1 = L \subseteq V$ of dimension $n-d$ such that $B_2 \subseteq A_2$ has dimension $(n-d+1)(n-d)/2$. Given a subspace $L \subseteq V$ we will write $k\langle L\rangle$ for the subalgebra generated by L.

Consider the linear system $\mathbb{P}(W) \subseteq |\mathcal{O}_{\mathbb{P}(V)}(2)|$. Let L be a subspace of V. We can restrict the quadratic functions in $W \subseteq \operatorname{Sym}^2 V^*$ to L and we obtain the following composition of maps

$$W \hookrightarrow \operatorname{Sym}^2 V^* \to \operatorname{Sym}^2 L^*.$$

Let M_L be the kernel of this map, the functions in M that vanish on $\mathbb{P}(L)$. So $\mathbb{P}(M_L)$ is the linear system of quadrics in W that contain $\mathbb{P}(L)$. So we get an induced system of quadrics $\mathbb{P}(W/M_L) \subseteq |\mathcal{O}_L(2)|$. The members of this system are of the form $Q_w \cap L$ where $w \neq 0$ in W/M_L. If p is a base point of the system $\mathbb{P}(W/M_L)$ then $p \in Q_w \cap \mathbb{P}(L)$ for all $w \neq 0$ in W/M_L and clearly $p \in Q_m$ for $m \in Q_m$ since $Q_m \supset \mathbb{P}(L)$. Hence we have the following statement which we record for later use.

Proposition 3.1 *Let W be a linear system of quadrics on $\mathbb{P}(V)$ and let L be a subspace of V. Let $\mathbb{P}(W/M_L)$ be the induced system of quadrics $Q \cap \mathbb{P}(L)$ in $\mathbb{P}(L)$. Then the base points of $\mathbb{P}(W/M_L)$ are contained in the base points of W, so $\mathrm{Bs}\,|Q \cap \mathbb{P}(L)| \subseteq \mathrm{Bs}\,|Q|$. Also*

$$\dim \mathrm{Bs}\,|\mathbb{Q} \cap \mathbb{P}(L)| + 1 \geq \dim L - \dim W + \dim M_L.$$

Proof. The first part follows from the observations above and the inequality follows from the fact that the base locus is the intersection of $\dim(W/M_L)$ many quadrics in $\mathbb{P}(L)$. □

Corollary 3.2 *Suppose the linear system of quadrics W is base point free, then any induced linear system is also base point free. Also*

$$\dim W \geq \dim L + \dim M_L.$$

So in particular if W is base point free of dimension n then

$$\mathcal{L}(A, d) := \{L \in \mathbb{G}(n-d, V) : \dim M_L \geq d\} = \{L \in \mathbb{G}(n-d, V) : \dim M_L = d\}$$

is a closed subvariety of $\mathbb{G}(n-d, V)$.

Next important observation is the following. Let L be a subspace of V. Let $I \subset V \otimes V$ be the relations A.

Proposition 3.3 *Let $A = k\langle V\rangle/R$ be a graded Clifford algebra. Let L be a linear subspace of V. Then $V \otimes L \cap R = L \otimes V \cap I = L \otimes L \cap R$.*

Proof. This follows from the fact that $R \subseteq \mathrm{Sym}^2 V$, and

$$L \otimes V \cap \mathrm{Sym}^2 V = V \otimes L \cap \mathrm{Sym}^2 V = L \otimes L \cap \mathrm{Sym}^2 V.$$

□

The following facts are well known from the theory of Koszul Algebras.

$$A_2^{!*} = R$$

$$A_3^{!*} = R \otimes V \cap V \otimes R$$

$$\cdots$$

$$A_i^{!*} = \bigcap V \otimes \cdots \otimes V \otimes R \otimes V \otimes \cdots \otimes V \subseteq V^{\otimes i}$$

Proposition 3.4 *There is a linear resolution of k_A given by $(A_i^{!*} \otimes A)_i$ where the differential is given by*

$$\begin{array}{rcl} A_i^{!*} \otimes A & \to & A_{i-1}^{!*} \otimes A \\ v_1 \otimes \cdots \otimes v_i \otimes a & \mapsto & v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i a. \end{array}$$

Write $k\langle L\rangle$ for the subalgebra of A generated by L.

Proposition 3.5 *Let L be a linear subspace of V that generates a regular subalgebra. Then the resolution above restricts to a resolution of A/LA. So there is a natural linear resolution $(k\langle L\rangle_i^{!*} \otimes A)_i$ with differential the restriction of the above to*

$$L \otimes \cdots \otimes L \otimes R_L \otimes L \otimes \cdots \otimes L.$$

where R_L denotes $R \cap L \otimes L$.

The following Theorem was proved in [13] for the case $n = 4, d = 1$.

Theorem 3.6 *The scheme of linear modules of dimension d of the algebra A is naturally isomorphic to the scheme*

$$\mathcal{L}(A, d) := \{L \in \mathbb{G}(n-d, V) : \dim\{w \in W : Q_w \supset L\} \geq d\}$$

and is also naturally isomorphic to the scheme of $n-d$ dimensional regular subalgebras of A. We have isomorphisms given by $L \mapsto A/LA$ and $L \mapsto k\langle L\rangle$.

We can restate the above Theorem geometrically by saying that it is always possible to project away from a linear module to a linear space, and we have a bijection between linear modules and linear projections.

An interesting consequence of this Theorem is that we can pull back linear modules via a projection to obtain linear modules in the original space and so we can reduce dimensions by projection when we study linear modules.

Theorem 3.7 *Let $k\langle L\rangle$ be a linear projection of A of dimension $n-d$. Let $M = k\langle L\rangle/I$ be a linear module of dimension k. Then A/IA is a linear module of A of dimension $d+k$, and the scheme of all linear modules of A of dimension $d+k$ that have A/LA as a quotient is isomorphic to the scheme of k dimensional linear modules of $k\langle L\rangle$*

The question of existence of d dimensional linear modules is not particularly easy, but it is fairly easy to see that existence is easy for small codimension and that when the codimension is large and the algebra A is generic there will not be any linear modules.

Theorem 3.8 *Let A be a generic regular graded Clifford algebra of dimension n. If*

$$n \leq \binom{n-2d+1}{2}$$

then $\mathcal{L}(A,d) = \emptyset$.

However there are always linear modules of low codimension. In particular there is an interesting variety of linear modules of codimension two. The proof of the following Theorem follows the same argument as the proof that the Reye Congruence is an Enriques surface, discussed in more detail in the next section.

Theorem 3.9 *Let A be a regular graded Clifford algebra of dimension n then $\mathcal{L}(A, n-2)$ has an étale double cover which is given by the intersection of n hyperplanes in the Segre embedding of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$*

If A is generic $\mathcal{L}(A, n-2)$ is a smooth projective variety of dimension $n-2$ and its double cover is a Calabi-Yau variety.

The following Theorem is easy to verify for $n = 3, 4$ on a computer. For general n, detailed calculations of the Hochschild cohomology are necessary using [15].

Theorem 3.10 *If A is a generic regular Clifford Algebra of dimension $n \geq 4$ then the degree zero components of the following cohomology groups are equal*

$$HH^2(A,A)_0 = Ext^2_{A^! \otimes A^!}(A^!, A^!)_0 = \mathrm{Ext}^1_A(\Omega^1, A)_0.$$

The above Theorem shows that any graded deformation of a generic $A^!$ is still commutative. Since graded deformations of $A^!$ are isomorphic to graded deformations of A by [5], [8], we see that any graded deformation of A is still a regular graded Clifford algebra. So we get the following:

Corollary 3.11 *The locus of regular graded Clifford Algebras of dimension $n \geq 4$ form a component of the moduli space of regular algebras of dimension n.*

4 Reye Congruence

In this section we review well known facts about the Reye Congruence associated to a web of quadrics in $\mathbb{P}^3$. The main results we will refer to are in Chapter VIII, 19, 20 and Ex. 14. in [4]. Let V be a four dimensional vector space and unless other wise stated, $\mathbb{P}^3 = \mathbb{P}(V)$. We start off with a divisor of bidegree $(1, 2)$ in $\mathbb{P}^3 \times \mathbb{P}^3$, which we will view as a web (linear system of projective dimension three) of quadrics in $\mathbb{P}^3$.

We will assume that the system has no base points. Then there are the following varieties associated to the web of quadrics.

Proposition 4.1 (The Reye Congruence) *Let $\mathbb{P}^3(W) = |Q_w|$ be a web of quadrics in $\mathbb{P}^3(V)$ with no base points and associated symmetric matrices A_w for $w \in W$.*

- *Let Δ be the points $p \in \mathbb{P}^3$ where Q_p is singular.*
- *Let X be the points (x, y) in $\mathbb{P}^3 \times \mathbb{P}^3$ with $xA_w y = 0$ for all w in $\mathbb{P}(W)$.*
- *Let Z be the points $q \in \mathbb{P}^3$ such that $q \in \operatorname{Sing} Q_p$ for some quadric Q_p in the web.*
- *Let E be the variety of lines ℓ in $\mathbb{G}(1, 3)$ that are contained in a pencil, i.e. the lines ℓ such that $\{p \in \mathbb{P}^3 | \ell \subset Q_p\}$ is a line in $\mathbb{P}^3$.*

Then the varieties X, Z, Δ are all surfaces with birational morphisms

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & & \\ \Delta & & \end{array}$$

The map $X \to Z$ is induced by choosing one of the two projections $\mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3$. There is an unramified degree two cover $X \to E$ induced by the mapping *the point* $(x,y) \in \mathbb{P}^3 \times \mathbb{P}^3$ *to the line* $\overline{xy}$*. The singular locus of* E *is given by the lines* ℓ *that are in a vertex of a quadric* Q_w *and also contained in another quadric of* $\mathbb{P}(W)$*, and the singular locus of* X *is determined similarly. If we assume the above singular locus is empty then*

- X *is a K3 surface and* E *is an Enriques surface with canonical double cover* X*.*
- Δ *is a quartic in* $\mathbb{P}^3$ *called a quartic symmetroid and has 10 ordinary double point singularities. The K3 surface* X *is a resolution of singularities of* Δ*.*
- *The singular locus of* Z *corresponds to the lines in* Δ*.*

Theorem 4.2 *The centre of* A *is the Fano three fold* Z*, which is the double cover of* $\mathbb{P}^3$ *ramified on the quartic symmetroid* Δ*. It has 10 ordinary double point singularities and* $\operatorname{Proj} A$ *is Azumaya away from these points. Etale locally at the singular points,* A *is an Atiyah flop algebra as in the [14].*

Theorem 4.3 *The point scheme of* A *is a double cover of the 10 ordinary double points of the quartic symmetroid* Δ*.*

Theorem 4.4 *The line scheme of* A *is the surface* S*, and so is a Enriques surface when smooth.*

5 Examples

We will study two degenerate examples.

5.1 20 points come together

This example is studied in [11]. We let w, x, y, z be a basis of W and let W, X, Y, Z be a basis of the $\mathbb{P}^3$ which contains the quadrics. Consider the web of quadrics defined by

$$A = \begin{pmatrix} 2w & z & x & y \\ z & 2x & 0 & 0 \\ x & 0 & 2y & 0 \\ y & 0 & 0 & 2z \end{pmatrix}.$$

This web is base point free, so we may apply the above analysis. The determinant of A is $16wxyz - 4xy^3 - 4x^3z - 4yz^3$ which cuts out the surface Δ. This surface is singular at the point $(1, 0, 0, 0)$. This only lines on this surface are given by $V(x, y) \cup V(y, z) \cup V(x, z)$, which meet at the singular point of Δ.

There is a unique quadric with rank one which lies above this singular point and is cut out by W^2. So any other quadric in the web that meets this quadric in lines, not a conic will give a singular point of the line scheme E. These are the quadrics that have rank two when restricted to the plane $W = 0$ which lie above $xyz = 0$. Hence E is singular along a triangle.

5.2 Toric Example

In this section we study the easy example of a web of quadrics determined by

$$A = \begin{pmatrix} 2w & 0 & 0 & 0 \\ 0 & 2x & 0 & 0 \\ 0 & 0 & 2y & 0 \\ 0 & 0 & 0 & 2z \end{pmatrix}.$$

This example is particularly easy to work with. It is immediate that Δ is given the union of the planes in a coordinate tetrahedron. The surface X is the intersection of four divisors of bidegree $(1, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$. This is given by four divisors isomorphic to $\mathbb{P}^2 \times \mathbb{P}^3 \cup \mathbb{P}^3 \times \mathbb{P}^2$. The result is a union of 8 projective planes and 6 smooth quadrics surfaces arranged in a cuboctahedron, where triangles are used to represent the planes and squares represent the quadrics. The line scheme E is a quotient of X by an involution and is given by the union of 4 projective planes and 3 smooth quadrics surfaces arranged in a cellular decomposition of the real projective plane. The four projective planes correspond the the lines on the faces of the coordinate tetrahedron, and the quadric surfaces parametrize the lines that meet two skew lines on the coordinate tetrahedron.

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